

Prime ideals in decomposable lattices

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Abstract. A distributive lattice L with minimum element 0 is called decomposable lattice if a and b are not comparable elements in L there exist $\bar{a}, \bar{b} \in L$ such that $a = \bar{a} \vee (a \wedge b)$, $b = \bar{b} \vee (a \wedge b)$ and $\bar{a} \wedge \bar{b} = 0$. The main purpose of this paper is to investigate prime ideals, minimal prime ideals and special ideals of a decomposable lattice. These are keys to understand the algebraic structure of decomposable lattices.

Key Words: decomposable lattice, prime ideal, minimal prime ideal, special ideal.

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1. Introduction

In [10] Grätzer and Schmidt characterized a Stone lattice as a distributive pseudocomplemented lattice in which every prime ideal contains a unique minimal prime ideal. Motivated by this characterization of Stone lattices, Cornish and Pawar characterized distributive lattices with minimum element 0 in which each prime ideal contains a unique minimal prime ideal (see e.g. [4,13]) and distributive lattices with 0 in which each prime ideal contains n minimal prime ideals [5]. They called such lattices respectively normal lattices and n -normal lattices. As a natural generalization of normal

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lattices, Cornish also introduced the concept of relatively normal lattices: a relatively normal lattice is a distributive lattice with 0 such that every bound closed interval is a normal lattice [5]. Filipoiu and Georgescu investigated values (regular ideals) in relatively normal lattices [7]. Hart, Snodgrass and Tsinakis further studied the structure of relatively normal lattices (see e.g. [11,14,15]). Motivated by the above works, we shall be concerned with decomposable lattices by replacing the "normality" by the "decomposability", i.e., a decomposable lattice is a distributive lattice L with minimum element 0 such that for any $a, b \in L$, if a and b are not comparable elements, written by $a \parallel b$, then there exist $\bar{a}, \bar{b} \in L$ such that $a = \bar{a} \vee (a \wedge b)$, $b = \bar{b} \vee (a \wedge b)$ and $\bar{a} \wedge \bar{b} = 0$.

Decomposability is not just the algebraic properties for some lattices. There exist in other algebraic areas, such as rings, modules and lattice-ordered group. We will see examples in section 2. Decomposable lattice is the common tool to understand these properties. Furthermore, the characterizations of prime ideals, minimal prime ideals and special ideals in the decomposable lattice are explicit. More details will be seen in later. Moreover, these characterizations can be our main technical tool for the further study of the structure of such lattices. In fact, with the help of the results of the present paper, the structure of decomposable lattices determined by their prime ideals, minimal prime ideals and special ideals can be developed [12].

Here is a brief outline of the article. In Section 1, we simply review some basic definitions and some well-known results. Three examples of decomposable lattices in lattices, rings and lattice-ordered groups, respectively are given. In Section 2, we investigate prime ideals of a decomposable lattice and the relationship between prime ideals and regular ideals. This is contained in Section 3, where we shall first establish explicit characterizations of minimal prime ideals of a decomposable lattice and then investigate the relationship among prime ideals, minimal prime ideals and regular ideals. We investigate special ideals of a decomposable lattice and the relationship between special ideals and regular ideals in the last section.

2. Preliminaries and Examples

Firstly, we simply review some basic definitions and some well-known results. The reader is referred to [9] for the general theory of lattices.

Throughout this paper, we consider lattices L with minimum element 0, denote by \mathbb{DL} the class of decomposable lattices and use " \subset " and " \supset " to denote proper set-inclusion.

A lattice L is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$. A nonempty subset I in a lattice L is called an ideal of L if $a \vee b \in I$ for any $a, b \in I$ and $a \geq x \in L$ implies that $x \in I$. We denote by $Ide(L)$ the set of all ideals of L . In particular, if $a \in L$ then $(a] = \{x \in L \mid x \leq a\}$ is called the principal ideal of L generated by a . A direct computation shows that if $L \in \mathbb{DL}$ then $Ide(L)$ is a distributive lattice by the rule: $I \wedge J = I \cap J$ and $I \vee J = \{a \vee b \mid a \in I, b \in J\}$ for any $I, J \in Ide(L)$.

An ideal P in a lattice L is called prime if $P \neq L$ and $a \wedge b \in P$ implies that either $a \in P$ or $b \in P$, where $a, b \in L$. By Zorn's Lemma, each prime ideal contains a minimal prime ideal. We denote by $Spe(L)$ and $MinSpe(L)$ respectively the set of all prime ideals of L and the set of all minimal prime ideals of L .

Let L be a lattice. For any $0 < x \in L$, by Zorn's Lemma, there exists a maximal ideal of L with respect to not containing x , denoted M , M is called a regular ideal and is the value of x . In general, a need not have a unique value. We denote by $Val(x)$ the set of all values of x . If M is the unique value of x , M or x is called special. We denote by $V(L)$ and $S(L)$ respectively the set of all values of L and the set of all special values of L . Clearly, $S(L) \subseteq V(L)$. Observe that the following conditions are equivalent: (1) $M \in V(L)$; (2) M is meet-irreducible, i.e., if $\bigcap_{\lambda \in \Lambda} I_\lambda = M$, where $\{I_\lambda\}_{\lambda \in \Lambda} \subseteq Ide(L)$, then $I_\lambda = M$ for some λ ; (3) $M \subset M^* = \bigcap \{I \in Ide(L) \mid I \supset M\}$; (4) $M \in Val(x)$, where $x \in M^* \setminus M$.

For a lattice L and $\emptyset \neq A \subseteq L$, we write $A^\perp = \{x \in L \mid x \wedge a = 0 \text{ for any } a \in A\}$. A^\perp is called the polar of A , and define $(A^\perp)^\perp = A^{\perp\perp}$. $P \in Ide(L)$ is called polar if $P = A^\perp$ for some $\emptyset \neq A \subseteq L$. Clearly, $P \in Ide(L)$ is polar if and only if $P = P^{\perp\perp}$. We denote by $P(L)$ the set of all polar ideals of L .

Let L be a lattice. A nonempty subset F of L is called a filter of L if the following conditions are satisfied: (1) $0 \notin F$; (2) for any $a, b \in F$, $a \wedge b \in F$; (3) if $x \in L$ and $x \geq a \in F$ implies $x \in F$. By Zorn's Lemma, each filter F of L must be contained in a maximal filter U of L , and U is called an ultrafilter of L .

We give the definition of decomposable lattice as following.

Definition 2.1. A decomposable lattice is a distributive lattice L with minimum element 0 such that for any $a, b \in L$, if $a \parallel b$ then there exist $\bar{a}, \bar{b} \in L$ such that $a = \bar{a} \vee (a \wedge b)$, $b = \bar{b} \vee (a \wedge b)$ and $\bar{a} \wedge \bar{b} = 0$.

Followings are examples of decomposable lattices, which are closely related to rings and lattice-ordered groups as well as lattices.

Recall that a lattice L is called strongly projectable if $L = (a] \vee a^\perp$ for any $a \in L$.

Example 2.2. Let L be a distributive lattice. If L is strongly projectable then $L \in \mathbb{DL}$.

Proof. Given any $a, b \in L$ with $a \parallel b$, since $L = (a \wedge b] \vee (a \wedge b)^\perp$, there exist $x_1, x_2 \in (a \wedge b]$ and $y_1, y_2 \in (a \wedge b)^\perp$ such that

$$a = x_1 \vee y_1, \quad b = x_2 \vee y_2.$$

Since L is distributive, we have

$$a \wedge b = (a \wedge b) \wedge (x_1 \vee y_1) = (a \wedge b \wedge x_1) \vee (a \wedge b \wedge y_1) = a \wedge b \wedge x_1.$$

So $a \wedge b \leq x_1$, which implies that $x_1 = a \wedge b$. Similarly, $x_2 = a \wedge b$. Then

$$a = a \wedge (x_1 \vee y_1) = (a \wedge x_1) \vee (a \wedge y_1) = (a \wedge y_1) \wedge (a \wedge b).$$

Similarly, $b = (b \wedge y_2) \wedge (a \wedge b)$. Since

$$(a \wedge y_1) \wedge (b \wedge y_2) = (a \wedge b) \wedge y_1 \wedge y_2 = 0,$$

we get $L \in \mathbb{DL}$.

Recall from [2,6] that a partially ordered group is both a group $(G, +)$ and a partially ordered set (G, \leq) whenever $a \leq b$ and $x, y \in G$ then $x + a + y \leq x + b + y$. A lattice-ordered group is a partially ordered group G and the underlying order is a lattice. A lattice-ordered group is called complete if every subset bounded above has a least upper bound and every subset bounded below has a greatest lower bound [3]. Recall also from [1] that a lattice-ordered group G is called compactly generated if $\{a_\lambda\}_{\lambda \in \Lambda}$ is a nonempty subset of L and $\bigwedge_{\lambda \in \Lambda} a_\lambda = 0$ then there exists a finite subset $\{a_i\}_{i=1}^n$ of $\{a_\lambda\}_{\lambda \in \Lambda}$ such that $\bigwedge_{i=1}^n a_i = 0$.

Example 2.3. Let $(G, +, \vee, \wedge)$ be a complete lattice-ordered group. If G is compactly generated then the positive cone $G^+ = \{x \in G \mid x \geq 0\} \in \mathbb{DL}$.

Proof. By hypothesis, each positive element in G can be written as a join of some atoms in G . So, for any $x, y \in G^+$ with $x \parallel y$, write

$$x = \bigvee_{\lambda \in \Lambda_1} a_\lambda, \quad y = \bigvee_{\mu \in \Lambda_2} b_\mu,$$

where each a_λ and b_μ are atoms in G . If $\Lambda = \Lambda_1 \cap \Lambda_2$, then $x \wedge y = \bigvee_{\nu \in \Lambda} c_\nu$. Now, set

$$x' = \bigvee_{\lambda \in \Lambda_1 \setminus \Lambda} a_\lambda, \quad y' = \bigvee_{\mu \in \Lambda_2 \setminus \Lambda} b_\mu.$$

Then $x = x' \vee (x \wedge y)$, $y = y' \vee (x \wedge y)$. In view of [6], G is completely distributive, we further have

$$x' \wedge y' = \left(\bigvee_{\lambda \in \Lambda_1 \setminus \Lambda} a_\lambda \right) \wedge \left(\bigvee_{\mu \in \Lambda_2 \setminus \Lambda} b_\mu \right) = \bigvee_{\lambda \in \Lambda_1 \setminus \Lambda} \bigvee_{\mu \in \Lambda_2 \setminus \Lambda} (a_\lambda \wedge b_\mu) = 0.$$

So $G^+ \in \mathbb{DL}$.

Following Fuchs [8], a ring R is called arithmetical if the lattice $Ide(R)$ of all ideals in R is distributive, i.e., $I \cap (J + K) = (I \cap J) + (I \cap K)$ for any $I, J, K \in Ide(R)$.

Example 2.4. If R is an arithmetical ring and satisfies that for any $I \in Ide(R)$ there exists some $e^2 = e \in R$ such that $I = eR$, then $Ide(R) \in \mathbb{DL}$.

Proof. Given any $I, J \in Ide(R)$, if $I \parallel J$, write $K = I \cap J \in Ide(R)$, then there exists some $e^2 = e \in R$ such that $K = eR$. Since $I \subseteq R = eR \oplus (1 - e)R$, there exist $I_1, I_2 \in Ide(R)$ with $I_1 \subseteq eR, I_2 \subseteq (1 - e)R$ such that $I = I_1 + I_2$. Similarly, there exist $J_1, J_2 \in Ide(R)$ with $J_1 \subseteq eR, J_2 \subseteq (1 - e)R$ such that $J = J_1 + J_2$. Thus, we have

$$K = K \cap I = K \cap (I_1 + I_2) = (K \cap I_1) + (K \cap I_2) = K \cap I_1,$$

and hence $K \subseteq I_1$, so that $K = I_1$. Similarly, $K = J_1$. So

$$I = I \cap (I_1 + I_2) = (I \cap I_1) + (I \cap I_2) = (I \cap I_2) + (I \cap J)$$

and

$$J = J \cap (J_1 + J_2) = (J \cap J_1) + (J \cap J_2) = (J \cap J_2) + (I \cap J).$$

Write $I' = I \cap I_2, J' = J \cap J_2$. Then $I = I' + (I \cap J), J = J' + (I \cap J)$ and

$$I' \cap J' = (I \cap I_2) \cap (J \cap J_2) = (I \cap J) \cap (I_2 \cap J_2) \subseteq eR \cap (1 - e)R = 0.$$

Therefore $Ide(R) \in \mathbb{DL}$.

3. Prime ideals

In this section, we shall first establish characterizations of prime ideals of a decomposable lattice and then investigate the relationship between prime ideals and regular ideals.

Theorem 3.1. Let $L \in \mathbb{DL}$ and $L \neq P \in Ide(L)$. The following conditions are equivalent:

- (1) $P \in Spe(L)$.
- (2) If $x \wedge y = 0$ then either $x \in P$ or $y \in P$ for $x, y \in L$.
- (3) $x, y \in L \setminus P$ implies $x \wedge y \in L \setminus P$.
- (4) If $I \cap J \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$ for $I, J \in Ide(L)$.
- (5) If $I, J \in Ide(L)$ such that $P \subseteq I$ and $P \subseteq J$, then either $I \subseteq J$ or $J \subseteq I$.

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) is clear.

(1) \Rightarrow (4) Let $I, J \in Ide(L)$ be such that $I \cap J \subseteq P$. If $I \not\subseteq P$, then we may pick $a \in I \setminus P$. So, for any $b \in J$, since $a \wedge b \in I \cap J \subseteq P$ and $a \notin P$, we get that $b \in P$, so that $J \subseteq P$.

(4) \Rightarrow (1) Given $a, b \in L$ with $a \parallel b$, if $a \wedge b \in P$, then $(a \wedge b] \subseteq P$. Notice that $(a \wedge b] = (a] \cap (b]$. Then $(a] \cap (b] \subseteq P$. So, by (4), we get that either $(a] \subseteq P$ or $(b] \subseteq P$, hence either $a \in P$ or $b \in P$. Therefore $P \in Spe(L)$.

(1) \Rightarrow (5) Let $I, J \in Ide(L)$ be such that $P \subseteq I$ and $P \subseteq J$. Suppose that I and J are not comparable, written $I \parallel J$. Pick $a \in I \setminus J, b \in J \setminus I$. Clearly, $a \parallel b$. Then there exist $\bar{a}, \bar{b} \in L$ such that $a = \bar{a} \vee (a \wedge b), b = \bar{b} \vee (a \wedge b)$ and $\bar{a} \wedge \bar{b} = 0 \in P$. So either $\bar{a} \in P$ or $\bar{b} \in P$, which implies that either $a \in J$ or $b \in I$, a contradiction.

(5) \Rightarrow (1) Given $a, b \in L$ with $a \wedge b \in P$, then $(a] \vee P, (b] \vee P \supseteq P$. By (5), $(a] \vee P$ and $(b] \vee P$ are comparable. Without loss of generality, assume that $(a] \vee P \subseteq (b] \vee P$. Since $Ide(L)$ is a distributive lattice, we then have

$$P = (a \wedge b] \vee P = ((a] \cap (b]) \vee P = ((a] \vee P) \cap ((b] \vee P) = (a] \vee P.$$

Hence $a \in P$. Therefore $P \in Spe(L)$. □

By Theorem 3.1, we now get some immediate corollaries which should demonstrate some of the importance of prime ideals.

Corollary 3.2. Let $L \in \mathbb{DL}$.

- (1) $V(L) \subseteq Spe(L)$.
- (2) $\bigcap V(L) = \bigcap Spe(L) = 0$.
- (3) For any $I \in Ide(L)$, $I = \bigcap \{M \in V(L) \mid M \supseteq I\} = \bigcap \{P \in Spe(L) \mid P \supseteq I\}$.

Corollary 3.3. Let $L \in \mathbb{DL}$.

- (1) The intersection of a chain of prime ideals of L is prime.
- (2) If $P \in \text{Spe}(L)$ then the set $\{I \in \text{Ide}(L) \mid I \supseteq P\}$ forms a chain.
- (3) L is totally ordered if and only if the zero ideal 0 of L is prime.

Corollary 3.4. Let $L \in \mathbb{DL}$. The following conditions are equivalent:

- (1) Each prime ideal of L contains a unique minimal prime ideal.
- (2) For any $N_1, N_2 \in \text{MinSpe}(L)$, if $N_1 \parallel N_2$ then $L = N_1 \vee N_2$.
- (3) For any $P_1, P_2 \in \text{Spe}(L)$, if $P_1 \parallel P_2$ then $L = P_1 \vee P_2$.
- (4) For any $M_1, M_2 \in V(L)$, if $M_1 \parallel M_2$ then $L = M_1 \vee M_2$.

As an application of Theorem 3.1, we now investigate the relationship between prime ideals and regular ideals of a decomposable lattice.

Theorem 3.5. Let $L \in \mathbb{DL}$. The following conditions are equivalent:

- (1) $\text{Spe}(L) = V(L)$.
- (2) $V(L)$ satisfies *DCC*.
- (3) $\text{Spe}(L)$ satisfies *DCC*.

Proof. (2) \Leftrightarrow (3) is clear. It suffices to show (1) \Leftrightarrow (2)

(1) \Rightarrow (2) Given any descending chain of $V(L)$: $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supseteq \cdots$, and set $P = \bigcap_{i=1}^{\infty} Q_i$. A direct computation shows that $P \in \text{Spe}(L)$, hence $P \in V(L)$ by hypothesis. Then there exists some positive integer m such that $P = Q_m$. So $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_m = Q_{m+1} = \cdots$. Thus $V(L)$ satisfies *DCC*.

(2) \Rightarrow (1) Given any $P \in \text{Spe}(L)$, then

$$P = \bigcap \{Q \in V(L) \mid Q \supseteq P\}.$$

Since P is prime, by Corollary 3.3, the set $\{Q \in V(L) \mid Q \supseteq P\}$ is a chain. Since $V(L)$ satisfies *DCC*, this chain is finite, and hence it must have a least element, denoted Q_0 , so that $P = Q_0 \in V(L)$. So $\text{Spe}(L) = V(L)$. \square

4. Minimal prime ideals

In this section, we first investigate the relationship between ultrafilters and minimal prime ideals in a decomposable lattice. With the help of the relationship, we shall establish explicit characterizations of minimal prime ideals of a decomposable lattice, which are pure lattice-theoretic extension of the corresponding results of lattice-ordered groups [2,6].

Filters arise naturally whenever we have a partially ordered set. We remind the reader that if L is a lattice and E is a \wedge -semilattice of L (i.e. for any $a, b \in E$, $a \wedge b \in E$) then $\overline{E} = \bigcap \{F \mid E \subseteq F \text{ a filter of } L\}$ is the smallest filter of L containing E , is called the filter of L generated by E . By Zorn's Lemma, each filter F of L must be contained in a maximal filter U of L , and U is called an ultrafilter of L .

Lemma 4.1. Let $L \in \mathbb{DL}$ and U a \wedge -semilattice with $0 \notin U$. The following conditions are equivalent:

- (1) U is an ultrafilter of L .
- (2) For any $x \in L \setminus U$ there exists $u \in U$ such that $x \wedge u = 0$.
- (3) $L \setminus U \in \text{MinSpe}(L)$.

Proof. (1) \Rightarrow (2) Assume that there exists some $x \in L \setminus U$ such that for any $u \in U$, $x \wedge u > 0$. A direct computation shows that the set

$$U_0 = \{x \wedge u \mid u \in U \cup \{x\}\}$$

is a \wedge -semilattice with $0 \notin U_0$. Let \overline{U} be the filter of L generated by U_0 . Then $U_0 \subseteq \overline{U}$, so that $U \subseteq \overline{U}$. But $x \in \overline{U}$ and $x \notin U$, which contradicts the fact that U is an ultrafilter of L .

(2) \Rightarrow (3) We first show that $L \setminus U \in \text{Ide}(L)$. Since U is a filter, it suffices to show that for any $x, y \in L \setminus U$, $x \vee y \in L \setminus U$. Assume that there exist $x, y \in L \setminus U$ such that $x \vee y \in U$. By (2), there exist $a, b \in U$ such that $x \wedge a = 0, y \wedge b = 0$. Set $c = a \wedge b \in U$. Then $0 = c \wedge (x \vee y) \in U$, a contradiction.

We next show that $L \setminus U \in \text{MinSpe}(L)$. Clearly, $L \setminus U \in \text{Spe}(L)$. Assume that $L \setminus U \notin \text{MinSpe}(L)$. Then by Zorn's Lemma, there exists some $M \in \text{MinSpe}(L)$ such that $L \setminus U \supset M$. Observe that $L \setminus M$ is a filter of L and clearly $L \setminus M \supset U$, which contradicts the fact that U is an ultrafilter of L . So $L \setminus U \in \text{MinSpe}(L)$.

(3) \Rightarrow (1) Clearly, if $L \setminus U \in \text{MinSpe}(L)$ then U is a filter of L . Assume that U is not an ultrafilter of L . Then there exists an ultrafilter of L , denoted W , such that $W \supset U$. Using the result of (1) \Rightarrow (2), $L \setminus W \in \text{MinSpe}(L)$. But $L \setminus W \supset L \setminus U \in \text{MinSpe}(L)$, a contradiction. Therefore U is an ultrafilter of L . \square

Lemma 4.2. Let $L \in \mathbb{DL}$. If X is a \wedge -semilattice with $0 \notin X$ then

$$\bigcup \{a^\perp \mid a \in X\} = \bigcap \{P \in \text{Spe}(L) \mid P \cap X = \emptyset\} = \bigcap \{M \in \text{MinSpe}(L) \mid M \cap X = \emptyset\}.$$

In particular, if $P \in \text{Spe}(L)$ then

$$\bigcup\{a^\perp \mid a \in L \setminus P\} = \bigcap\{M \in \text{MinSpe}(L) \mid M \subseteq P\}.$$

Proof. The second equation is clear. It suffices to show the first equation.

Clearly, $\bigcup\{a^\perp \mid a \in X\} \subseteq \bigcap\{P \in \text{Spe}(L) \mid P \cap X = \emptyset\}$. If $x \notin \bigcup\{a^\perp \mid a \in X\}$ then $x \notin a^\perp$ for any $a \in X$, i.e., $x \wedge a > 0$ for any $a \in X$. Consider the set

$$\overline{X} = \{x \wedge a \mid a \in X \cup \{x\}\}.$$

A direct computation shows that \overline{X} is a \wedge -semilattice with $0 \notin \overline{X}$. Let F be the filter of L generated by \overline{X} . Then there exists an ultrafilter U of L such that $U \supseteq F$. By Lemma 4.1, $P = L \setminus U \in \text{MinSpe}(L)$ and $P \cap X = \emptyset$. Since $x \in U$, we get $x \notin P$, so that $x \notin \bigcap\{P \in \text{Spe}(L) \mid P \cap X = \emptyset\}$. So $\bigcup\{a^\perp \mid a \in X\} = \bigcap\{P \in \text{Spe}(L) \mid P \cap X = \emptyset\}$.

Using the above results, the remains are clear. \square

Now we can apply Lemma 4.1 and Lemma 4.2 to establish characterizations of minimal prime ideals of a decomposable lattice.

Theorem 4.3. Let $L \in \mathbb{DL}$ and $P \in \text{Spe}(L)$. The following conditions are equivalent:

- (1) $P \in \text{MinSpe}(L)$.
- (2) $P = \bigcup\{a^\perp \mid a \notin P\}$.
- (3) For any $x \in P$, $x^\perp \not\subseteq P$.

Proof. (1) \Rightarrow (2) By Lemma 4.2, we have

$$\bigcup\{a^\perp \mid a \notin P\} = \bigcap\{M \in \text{MinSpe}(L) \mid M \subseteq P\}.$$

Since $P \in \text{MinSpe}(L)$, this means that the set $\{M \in \text{MinSpe}(L) \mid M \subseteq P\} = \{P\}$, so $P = \bigcup\{a^\perp \mid a \notin P\}$.

(2) \Rightarrow (3) By (2),

$$P = \bigcup\{a^\perp \mid a \notin P\}.$$

So, for any $x \in P$, there exists some $a \notin P$ such that $x \in a^\perp$. Then $a \in x^\perp$, which implies $a \in x^\perp \setminus P$. Therefore $x^\perp \not\subseteq P$.

(3) \Rightarrow (1) Assume that $P \notin \text{MinSpe}(L)$. Then there exists some $M \in \text{MinSpe}(L)$ such that $P \supset M$. Pick $x \in P \setminus M$. Then for any $0 < y \in x^\perp$, $x \wedge y = 0 \in M$. Since M is prime and $x \notin M$, we get $y \in M$, and hence $x^\perp \subseteq M \subset P$, a contradiction. So $P \in \text{MinSpe}(L)$. \square

We now apply Theorem 4.3 to investigate the relationship among prime ideals, minimal prime ideals and regular ideals. In order to do this, we need the following two lemmas.

Lemma 4.4. Let $L \in \mathbb{DL}$ and $0 \neq A \in Ide(L)$. Then

$$A^\perp = \bigcap \{M \in Spe(L) \mid A \not\subseteq M\} = \bigcap \{P \in MinSpe(L) \mid A \not\subseteq P\}.$$

Proof. It suffices to show the first equation.

If $A \not\subseteq M$, then pick $a \in A \setminus M$, so that $a^\perp \subseteq M$ since $M \in Spe(L)$. Hence $A^\perp \subseteq a^\perp \subseteq M$. So $A^\perp \subseteq \bigcap \{M \in Spe(L) \mid A \not\subseteq M\}$.

Now, suppose that $A^\perp \subset \bigcap \{M \in Spe(L) \mid A \not\subseteq M\}$. Pick $0 < b \in (\bigcap \{M \in Spe(L) \mid A \not\subseteq M\}) \setminus A^\perp$. Then there exists some $0 < c \in A$ such that $b \wedge c > 0$. Thus $b \wedge c \in A$. Now, Let $M \in Val(b \wedge c)$. Then $b \wedge c \notin M$. So $b \wedge c \notin \bigcap \{M \in Spe(L) \mid A \not\subseteq M\}$, which contradicts the fact that $b \in \bigcap \{M \in Spe(L) \mid A \not\subseteq M\}$. So $A^\perp = \bigcap \{M \in Spe(L) \mid A \not\subseteq M\}$. \square

Lemma 4.5. Let $L \in \mathbb{DL}$ and $a, b \in L \setminus \{0\}$. The following conditions are equivalent:

- (1) a and b are disjoint, i.e., $a \wedge b = 0$.
- (2) $Val(a) \cap Val(b) = \emptyset$ and $Val(a) \cup Val(b) = Val(a \vee b)$.

Proof. (1) \Rightarrow (2) Suppose that $Val(a) \cap Val(b) \neq \emptyset$. Then there exists $M \in Val(a) \cap Val(b)$ such that $a \notin M$ and $b \notin M$. So $a \wedge b \notin M$ implies $a \wedge b \neq 0$, a contradiction. Now, if $Q \in Val(a \vee b)$ then $a \vee b \notin Q$. Since Q is an ideal of L , we get that either $a \notin Q$ or $b \notin Q$. Without loss of generality, assume that $a \notin Q$. Then there exists some $Q_a \in Val(a)$ such that $Q \subseteq Q_a$. If $Q \subset Q_a$, then $a \vee b \in Q^* \subseteq Q_a$ (Q^* denotes the cover of Q in $Ide(L)$), so that $a \in Q_a$, a contradiction. So $Q = Q_a \in Val(a)$. Conversely, if $K \in Val(a) \cup Val(b)$ then either $K \in Val(a)$ or $K \in Val(b)$. Without loss of generality, assume that $K \in Val(a)$, then $a \notin K$, and hence $a \vee b \notin K$. So there exists some $Q \in Val(a \vee b)$ such that $K \subseteq Q$. If $K \subset Q$, then $a \in K^* \subseteq Q$. Since $b \in P \subset Q$, $a \vee b \in Q$, a contradiction. Therefore $P = Q \in Val(a \vee b)$.

(2) \Rightarrow (1) Suppose that $a \wedge b \neq 0$. Let $M \in Val(a \wedge b)$. Then $a \notin M$. So there exists some $P \in Val(a)$ such that $P \supseteq M$. Similarly, $b \notin M$. So there exists some $Q \in Val(b)$ such that $Q \supseteq M$. By Corollary 3.3, P and Q are comparable. Again, $Val(a) \cup Val(b) = Val(a \vee b)$, so that $P = Q$, which contradicts $Val(a) \cap Val(b) = \emptyset$. Therefore a and b are disjoint. \square

By induction on n , one can obtain that if $\{a_1, a_2, \dots, a_n\}$ is a mutually disjoint subset of L then $Val(\bigvee_{i=1}^n a_i) = \bigcup_{i=1}^n Val(a_i)$.

Theorem 4.6. Let $L \in \mathbb{DL}$ and $0 \neq I \in Ide(L)$. The following conditions are equivalent:

- (1) I is totally ordered.
- (2) For any $0 < a \in I$, $a^\perp = I^\perp$.
- (3) $I^\perp \in Spe(L)$.
- (4) $I^\perp \in MinSpe(L)$.
- (5) $I^{\perp\perp}$ is a maximal totally ordered ideal of L .
- (6) $I^{\perp\perp}$ is a minimal polar ideal of L .
- (7) I^\perp is a maximal polar ideal of L .
- (8) For any $0 < a \in I$, a is special.

Proof. (1) \Rightarrow (2) For any $0 < a \in I$, $a^\perp \supseteq I^\perp$ is clear. Assume that $a^\perp \supset I^\perp$. Pick $0 < x \in a^\perp \setminus I^\perp$. Then $x \wedge a = 0$ and $x \wedge b > 0$ for some $b \in I$. So $(x \wedge b) \wedge a = (x \wedge a) \wedge b = 0$. On the other hand, $0 < a, x \wedge b \in I$, and hence a and $x \wedge b$ are comparable, so that

$$(x \wedge b) \wedge a = \min\{x \wedge b, a\} > 0.$$

This is impossible. So $a^\perp = I^\perp$.

(2) \Rightarrow (3) By Theorem 3.1, it suffices to show that if $a, b \notin I^\perp$ then $a \wedge b \notin I^\perp$. Since $a \notin I^\perp$, there exists $0 < x \in I$ such that $a \wedge x > 0$. Similarly, $b \notin I^\perp$, there exists $0 < y \in I$ such that $b \wedge y > 0$. We claim that $(a \wedge x) \wedge (b \wedge y) > 0$. Otherwise, $(a \wedge x) \wedge (b \wedge y) = 0$, hence $b \wedge y \in (a \wedge x)^\perp = I^\perp$ by (2), so that $b \wedge y \in I \cap I^\perp = 0$, a contradiction. Therefore $I^\perp \in Spe(L)$.

(3) \Rightarrow (4) By Lemma 4.4, we have

$$I^\perp = \bigcap \{P \in MinSpe(L) \mid I \not\subseteq P\}.$$

Assume that $I^\perp \notin MinSpe(L)$. Then there exists some $P \in MinSpe(L)$ such that $I^\perp \supset P$, so that $I \subseteq P \subseteq I^\perp$. Thus $I = 0$, a contradiction. Thus $I^\perp \in MinSpe(L)$.

(4) \Rightarrow (5) We first show that $I^{\perp\perp}$ is totally ordered. Assume that there exist $0 < a, b \in I^{\perp\perp}$ such that $a \wedge b = 0$. Since I^\perp is prime, either $a \in I^\perp$ or $b \in I^\perp$, so that either $a = 0$ or $b = 0$, a contradiction.

We next show that $I^{\perp\perp}$ is maximal. Let J be a totally ordered ideal of L such that $J \supset I^{\perp\perp}$. Pick $0 < x \in J \setminus I^{\perp\perp}$. Then there exists some $0 < y \in I^\perp$ such that $x \wedge y > 0$. Now, pick $0 < a \in I$. Then $(x \wedge y) \wedge a = 0$ since $x \wedge y \in I^\perp$. On the other hand, $0 < x \wedge y \in J, a \in I \subseteq I^{\perp\perp} \subseteq J$ and J is totally ordered, so that $(x \wedge y) \wedge a = \min\{x \wedge y, a\} > 0$. This is impossible. Therefore $I^{\perp\perp}$ is a maximal totally ordered ideal of L .

(5) \Rightarrow (6) Let $D \in P(L)$ be such that $D \subseteq I^{\perp\perp}$. Then D is totally ordered. By using the result of (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5), $D = D^{\perp\perp}$ is a maximal totally ordered ideal of L , so that $D = I^{\perp\perp}$. So $I^{\perp\perp}$ is a minimal polar ideal of L .

(6) \Rightarrow (7) Since the map $P \rightarrow P^\perp$ for any $P \in P(L)$ is a dual isomorphism of lattices, $I^{\perp\perp}$ is a minimal polar ideal of L implies that I^\perp is a maximal polar ideal of L .

(7) \Rightarrow (8) For any $0 < a \in I$, assume that a has two distinct values Q_1 and Q_2 . Since $a \notin Q_1$ and $a \wedge b = 0$ for any $b \in I^\perp$, so that $I^\perp \subseteq Q_1$. Similarly, $I^\perp \subseteq Q_2$. Since Q_1 and Q_2 are incomparable, we may pick

$$0 < x \in Q_1 \setminus Q_2, 0 < y \in Q_2 \setminus Q_1 \text{ with } x \wedge y = 0.$$

So $x^\perp = y^\perp = I^\perp$ by the maximality of I^\perp . Again, $x \wedge y = 0$ implies that $x, y \in I^\perp \subseteq Q_1 \cap Q_2$, a contradiction. So a is special.

(8) \Rightarrow (1) Assume that I is not totally ordered. Then there exist $0 < a, b \in I$ such that $a \wedge b = 0$. By Lemma 4.5, $Val(a \vee b) = Val(a) \cup Val(b)$, i.e., $a \vee b$ has at least two distinct values, a contradiction. Therefore I must be totally ordered. \square

By using Theorem 4.6, we shall investigate the relationship between polar ideals and minimal prime ideals of a decomposable lattice.

Theorem 4.7. Let $L \in \mathbb{DL}$. If for any $P, Q \in P(L)$ either $L = P \vee Q$ or P and Q are comparable, then every polar ideal of L is minimal prime, i.e., $P(L) \subseteq MinSpe(L)$.

Proof. By way of contradiction. If there exists $P \in P(L)$ such that $P \notin MinSpe(L)$, write $P = A^\perp$, then A is not totally ordered by Theorem 4.6. So there exist $0 < a, b \in A$ such that $a \wedge b = 0$. We divide the proof into two steps.

Step 1. If a^\perp and b^\perp are incomparable then $L = a^\perp \vee b^\perp$ by hypothesis. Clearly, a^\perp and $a^{\perp\perp}$ are incomparable, then $L = a^\perp \vee a^{\perp\perp}$. So

$$a^{\perp\perp} = a^{\perp\perp} \cap L = a^{\perp\perp} \cap (a^\perp \vee b^\perp) = a^{\perp\perp} \cap b^\perp \subseteq b^\perp.$$

Similarly, $b^{\perp\perp} \subseteq a^\perp$.

(i) If $a^{\perp\perp}$ and $b^{\perp\perp}$ are incomparable then $L = a^{\perp\perp} \vee b^{\perp\perp}$. So

$$a^\perp = a^\perp \cap L = a^{\perp\perp} \cap (a^{\perp\perp} \vee b^{\perp\perp}) = a^\perp \cap b^{\perp\perp} \subseteq b^{\perp\perp}.$$

Thus $a^\perp = b^{\perp\perp}$. Similarly, $a^{\perp\perp} = b^\perp$. It follows that $a^\perp \cap b^\perp = a^\perp \cap a^{\perp\perp} = \{0\}$. So $A^\perp \subseteq a^\perp \cap b^\perp = \{0\}$, a contradiction.

(ii) If $a^{\perp\perp}$ and $b^{\perp\perp}$ are comparable then $a^{\perp\perp} \subseteq b^{\perp\perp} = a^\perp$, and hence $a^{\perp\perp} = 0$ or $b^{\perp\perp} \subseteq a^{\perp\perp} = b^\perp$, and hence $b^{\perp\perp} = 0$. It follows that $a^\perp = L$ or $b^\perp = L$, this is impossible.

Step 2. If a^\perp and b^\perp are comparable then $a^\perp \subseteq b^\perp$ or $b^\perp \subseteq a^\perp$. So $b \in a^\perp \subseteq b^\perp$ and hence $b = 0$ or $a \in b^\perp \subseteq a^\perp$ and hence $a = 0$, this is also impossible.

In view of Step 1 and Step 2, A is totally ordered. So $P \in \text{MinSpe}(L)$. \square

Recall that a lattice L is called projectable if $L = x^\perp \vee x^{\perp\perp}$ for any $x \in L$. We denote by \mathbb{T} the class of projectable lattices.

Theorem 4.8. Let $L \in \mathbb{DL}$. The following conditions are equivalent:

- (1) $\text{Spe}(L) = \text{MinSpe}(L)$.
- (2) $L = (a] \vee a^\perp$ for any $a \in L$.
- (3) $L \in \mathbb{T}$ and $(x] = x^{\perp\perp}$ for any $x \in L$.

Proof. (1) \Rightarrow (2) Assume that there exists $a \in L$ such that $(a] \vee a^\perp \subset L$. Pick $x \in L \setminus ((a] \vee a^\perp)$. Then there exists $M \in \text{Val}(x)$ such that $(a] \vee a^\perp \subseteq M$. By (1), M is minimal prime. But $a \in M$ and $a^\perp \subseteq M$, which contradicts Theorem 4.3.

(2) \Rightarrow (3) Clearly, $L \in \mathbb{T}$. Now, given any $x \in L$, $(x] \subseteq x^{\perp\perp}$ is clear. Again,

$$x^{\perp\perp} = x^{\perp\perp} \cap L = x^{\perp\perp} \cap ((x] \vee x^\perp) = (x^{\perp\perp} \cap (x]) \vee (x^{\perp\perp} \cap x^\perp) = x^{\perp\perp} \cap (x] \subseteq (x],$$

so $x^{\perp\perp} = (x]$.

(3) \Rightarrow (1) By (3), $L = (x] \vee x^\perp$ for any $x \in L$. Assume that there exists $P \in \text{Spe}(L)$ such that P is not minimal. Then there exists some $M \in \text{MinSpe}(L)$ such that $P \supset M$. Pick $a \in P \setminus M$. Then $a^\perp \subseteq M \subset P$, so that $L = (a] \vee a^\perp \subseteq P$, a contradiction. Therefore $\text{Spe}(L) = \text{MinSpe}(L)$. \square

Recall that a minimal element of a partially ordered set is an atom. If every element exceeds an atom, the partially ordered set is called atomic. Theorem 3.1 shows that the set $\text{Spe}(L)$ of all prime ideals of a decomposable lattice L is an atomic root system under inclusion. It is natural to ask under what condition to make $V(L)$ atomic, i.e., every regular ideal of L contains a minimal regular ideal. In order to do this, we need the following lemma. Since its proof is direct, we shall omit it.

Lemma 4.9. Let $L \in \mathbb{DL}$ and $M \in \text{Ide}(L)$. Then $M \in \text{MinSpe}(L)$ if and only if there exists a maximal chain $\{M_\lambda\}_{\lambda \in \Lambda}$ of $V(L)$ such that $M = \bigcap_{\lambda \in \Lambda} M_\lambda$.

Theorem 4.10. Let $L \in \mathbb{DL}$. If each prime ideal in L contains a finite number of minimal prime ideals then the following conditions are equivalent:

- (1) Every minimal prime ideal of L is regular, i.e., $MinSpe(L) \subseteq V(L)$.
- (2) $V(L)$ is atomic.

Proof. (1) \Rightarrow (2) is clear.

(2) \Rightarrow (1) Given any $P \in MinSpe(L)$, by Lemma 4.9, there exists a maximal chain of $V(L)$, write $\{Q_\gamma \in V(L) \mid \gamma \in \Delta\}$, such that $P = \bigcap_{\gamma \in \Delta} Q_\gamma$. Pick $Q_{\gamma_1} \in \Delta$. Since $V(L)$ is atomic, Q_{γ_1} contains an atom, write Q_1 . If $Q_\gamma \supseteq Q_1$ for any $\gamma \in \Delta$, then $P = Q_1$, we are done. Otherwise, there exists $Q_{\gamma_2} \in \Delta$ such that $Q_{\gamma_2} \subset Q_{\gamma_1}$, but $Q_1 \not\subseteq Q_{\gamma_2}$. Similarly, Q_{γ_2} contains an atom, write Q_2 . If $Q_\gamma \supseteq Q_2$ for any $\gamma \in \Delta$, then $P = Q_2$, we are done. We claim that this process must end. Otherwise, we may obtain an infinite number of atoms in $V(L)$, write $\{Q_1, Q_2, \dots, Q_n, \dots\}$ which satisfy $Q_i \neq Q_j$ for any $i \neq j$. Clearly, Q_{γ_1} contains each Q_i for all $i = 1, 2, \dots, n, \dots$, a contradiction. So each minimal prime ideal in L is regular. \square

5. Special ideals

In this section, we characterize special ideals of a decomposable lattice and then investigate the relationship among prime ideals, minimal prime ideals, regular ideals and special ideals.

Recall that for a lattice L and $0 < x \in L$, if M is the unique value of x , M or x is called special. We denote by $S(L)$ the set of all special ideals of the lattice L .

Theorem 5.1. Let L be a lattice and $M \in Ide(L)$. Then the following conditions are equivalent:

- (1) $M \in S(L)$.
- (2) If $\bigcap_{\lambda \in \Lambda} I_\lambda \subseteq M$, where $\{I_\lambda\}_{\lambda \in \Lambda} \subseteq Ide(L)$, then $I_\lambda \subseteq M$ for some λ .
- (3) M is the unique value of x , where $x \in M^* \setminus M$.

Proof. (1) \Leftrightarrow (2) is clear. It suffices to show (1) \Leftrightarrow (3)

(1) \Rightarrow (3) Consider the set

$$\Delta = \{J_\lambda \in Ide(L) \mid J_\lambda \not\subseteq M, \lambda \in \Lambda\}.$$

Since M is special, $0 \neq \bigcap_{\lambda \in \Lambda} J_\lambda \not\subseteq M$, and hence pick $0 \neq x \in (\bigcap_{\lambda \in \Lambda} J_\lambda) \setminus M$. Now, if K is an ideal of L with respect to not containing x , and $K \supset M$, then $K \in \Delta$, so that $x \in K$, a contradiction. So M is a maximal ideal with respect to not containing x . Again, if N is any maximal ideal of L with respect to not containing x , then since

$x \notin N$, $N \notin \Delta$. So $N \subseteq M$, and hence $N = M$. Therefore M is the unique maximal ideal of L with respect to not containing x and clearly $x \in M^* \setminus M$.

(3) \Rightarrow (1) Clearly, M is regular. Now, let $\{I_\lambda\}_{\lambda \in \Lambda}$ be any nonempty family of ideals of L such that $\bigcap_{\lambda \in \Lambda} I_\lambda \subseteq M$. Since $x \notin M$, there exists some $\lambda \in \Lambda$ such that $x \notin I_\lambda$. So there exists some $N \in Val(x)$ such that $I_\lambda \subseteq N$. By assumption, M is the unique maximal ideal with respect to not containing x , so that $N = M$. Therefore $I_\lambda \subseteq M$. So M is special. \square

In order to investigate the relationship among prime ideals, minimal prime ideals and special ideals of a decomposable lattice, we need the following two lemmas.

For a lattice L and $P \in Spe(L)$, write $S_P = \bigcap \{M \in MinSpe(L) \mid M \subseteq P\}$.

Lemma 5.2. Let $L \in \mathbb{DL}$ and $P_1, P_2 \in Spe(L)$. Then $S_{P_1} \subseteq P_2$ if and only if P_1 and P_2 are comparable.

Proof. The sufficiency is clear. For the necessity, assume that $P_1 \parallel P_2$. Pick $a_1 \in P_2 \setminus P_1, a_2 \in P_1 \setminus P_2$ with $a_1 \wedge a_2 = 0$. Now, let $M \in MinSpe(L)$ be such that $M \subseteq P_2$. Then $a_1 \in M$, so $a_1 \in S_{P_2}$. Similarly, $a_2 \in S_{P_1}$. Since $S_{P_1} \subseteq P_2$, we get $a_2 \in P_2$, a contradiction. \square

Lemma 5.3. Let $L \in \mathbb{DL}$ and $P \in Spe(L)$. Then $S_P = \{a \in L \mid a = 0 \text{ or for any } Q \in Spe(L) \text{ with } a \notin Q, Q \text{ and } P \text{ are not comparable}\}$.

Proof. Write $K = \{a \in L \mid a = 0 \text{ or for any } Q \in Spe(L) \text{ with } a \notin Q, Q \text{ and } P \text{ are not comparable}\}$. If $S_P \not\subseteq K$, pick $0 < a \in S_P \setminus K$, then there exists $Q \in Val(a)$ such that Q and P are comparable. If $Q \subseteq P$ then $a \in S_P \subseteq Q$ by Lemma 5.2, a contradiction. If $P \subset Q$ then $a \in S_P \subseteq P \subset Q$, a contradiction. So $S_P \subseteq K$. Conversely, if $K \not\subseteq S_P$, pick $0 < b \in K \setminus S_P$, then there exists $M \in MinSpe(L)$ with $M \subseteq P$ such that $b \notin M$. But $b \in K$, so that M and P are comparable, a contradiction. So $S_P = K$. \square

Theorem 5.4. Let $L \in \mathbb{DL}$ and $I \in Ide(L)$. The following conditions are equivalent:

- (1) There exists a unique value Q of g such that $Q \supseteq I$, and for any $x \in L \setminus I$, $x \wedge g \notin I$.
- (2) $S_P \subseteq I \subseteq P$, where $P \in Val(g)$.

Proof. (1) \Rightarrow (2) Let P be the unique value of g containing I . Since $I = \bigcap \{N \in V(L) \mid I \subseteq N\}$, it suffices to show that if $N \in V(L)$ with $N \supseteq I$ then $N \supseteq S_P$.

Suppose that there exists $N \in V(L)$ with $I \subseteq N$, but $S_P \not\subseteq N$. Then, by Lemma 5.2, $P \parallel N$. Pick $0 < x \in N^* \setminus N$ and $0 < y \in P \setminus N$. Then $0 < x \wedge y \in (N^* \setminus N) \cap P$. Using this method, we see that there exist

$$0 < a \in (N^* \setminus N) \cap P \text{ and } 0 < b \in (P^* \setminus P) \cap N.$$

Since $L \in \mathbb{DL}$, we may further assume that $a \wedge b = 0$. Now, if a has a value K such that $K \subseteq P$ then since $a \notin K$ implies $b \in K \subseteq P$, a contradiction. So each value of a is not comparable with P . By Lemma 5.3, $a \in S_P$. So $a \in (N^* \setminus N) \cap S_P$. Now, let $0 < x \in (N^* \setminus N) \cap S_P$. By (1), $x \wedge g \notin I$. Then there exists $K_{x \wedge g} \in Val(x \wedge g)$ such that $K_{x \wedge g} \supseteq I$. So $g \notin K_{x \wedge g}$, which implies that there exists some $K_g \in Val(g)$ such that $K_g \supseteq K_{x \wedge g} \supseteq I$. By (1), $K_g = P$. On the other hand, $x \notin K_{x \wedge g}$, there exists $K_x \in Val(x)$ such that $K_x \supseteq K_{x \wedge g}$, which implies that K_x and P are comparable. So $x \in S_P \subseteq K_x$, a contradiction. So $S_P \subseteq I \subseteq P$.

(2) \Rightarrow (1) Assume that there exists another value P_1 of g such that $S_P \subseteq I \subseteq P_1$. Note that $P_1 \neq P$ implies that $P \parallel P_1$. But, by Lemma 5.2, $S_P \subseteq P_1$ implies that P and P_1 are comparable, a contradiction. So $P = P_1$, and hence P is the only value of g containing I .

Now, let $\Gamma = \{M \in V(L) \mid I \subseteq M\}$. Clearly $I = \bigcap \{M \in V(L) \mid I \subseteq M\}$, which implies that M and P are comparable for any $M \in \Gamma$. Set $\Gamma_0 = \Gamma \setminus \{M \in V(L) \mid P \subset M\}$. Clearly $I = \bigcap_{M \in \Gamma_0} M$. Then for any $M \in \Gamma_0$, $M \subseteq P$. So if $x \notin I$ then there exists $M \in \Gamma_0$ such that $x \notin M$. Again, $g \notin P$, then $g \notin M$. So $x \wedge g \notin M$, and hence $x \wedge g \notin I$, as desired. \square

Theorem 5.5. Let $L \in \mathbb{DL}$ and $K \in Ide(L)$. The following conditions are equivalent:

- (1) $K \in Spe(L)$ and for any $x \in L \setminus K$, $x > K$.
- (2) $K \in Spe(L)$ and for any $I \in Ide(L)$, K and I are comparable.
- (3) For any $L \neq P \in P(L)$, $P \subseteq K$.
- (4) For any $M \in MinSpe(L)$, $M \subseteq K$.
- (5) For any $a \in L \setminus K$, $a^\perp = \{0\}$.
- (6) For any $a \in L \setminus K$, a is special.

Proof. (1) \Rightarrow (2) Assume that there exists some $I \in Ide(L)$ such that K and I are incomparable. Pick $x \in I \setminus K$. By (1), $x > K$, so that $K \subseteq I$, a contradiction.

(2) \Rightarrow (3) Given any $L \neq P \in P(L)$, if $P \not\subseteq K$ then $K \subset P$ by (2). Pick $x \in P \setminus K$. Since $K \in Spe(L)$ and $x \notin K$, $P^\perp \subseteq K \subset P$. So $P^\perp = 0$, and hence $P = P^{\perp\perp} = L$, a contradiction.

(3) \Rightarrow (4) For any $M \in MinSpe(L)$, by Theorem 4.3,

$$M = \bigcup \{a^\perp \mid a \notin M\}.$$

By (3), $M \subseteq K$.

(4) \Rightarrow (5) Given any $a \in L \setminus K$, by (4), $M \subseteq K$ for any $M \in \text{MinSpe}(L)$. Then $a \notin M$ for any $M \in \text{MinSpe}(L)$, so that $a^\perp \subseteq M$ for any $M \in \text{MinSpe}(L)$. So $a^\perp \subseteq \bigcap \text{MinSpe}(L) = 0$, i.e., $a^\perp = \{0\}$.

(5) \Rightarrow (6) Given any $a \in L \setminus K$, assume that a is not special. Then a has at least two distinct values Q_1, Q_2 . Clearly, $Q_1 \parallel Q_2$. Pick

$$a_1 \in Q_1 \setminus Q_2 \text{ and } a_2 \in Q_2 \setminus Q_1 \text{ with } a_1 \wedge a_2 = 0.$$

Clearly, $a_1, a_2 \notin K$, but $a_1^\perp \neq \{0\}$, a contradiction.

(6) \Rightarrow (1) We first show that $K \in \text{Spe}(L)$. Assume that there exist $0 < a, b \in L$ such that $a \wedge b = 0$, but $a \notin K$ and $b \notin K$. Then $a \vee b \notin K$. Notice that $a \wedge b = 0$, so that $\text{Val}(a \vee b) = \text{Val}(a) \cup \text{Val}(b)$. So $a \vee b$ is not special, a contradiction.

We next show that for any $x \in L \setminus K$, $x > K$. Otherwise, there exists $0 < k \in K$ such that $x \parallel k$. Since $L \in \mathbb{DL}$, we may further assume that $x \wedge k = 0$. Clearly, $x \vee k \notin K$. But $\text{Val}(x \vee k) = \text{Val}(x) \cup \text{Val}(k)$, it follows that $x \vee k$ is not also special, which ends the proof. \square

Recall that if $L \in \mathbb{DL}$ then the set $\text{Ide}(L)$ of all ideals of L is a distributive lattice by the rule: $I \wedge J = I \cap J$ and $I \vee J = \{a \vee b \mid a \in I, b \in J\}$. So $\text{Ide}(L)$ is α -distributive, i.e., for any $I \in \text{Ide}(L)$ and any subset $\{J_\lambda\}_{\lambda \in \Lambda} \subseteq \text{Ide}(L)$ with $|\Lambda| = \alpha$, $I \cap (\bigvee_{\lambda \in \Lambda} J_\lambda) = \bigvee_{\lambda \in \Lambda} (I \cap J_\lambda)$. But, in general, it is not dual α -distributive, i.e., $I \vee (\bigcap_{\lambda \in \Lambda} J_\lambda) = \bigcap_{\lambda \in \Lambda} (I \vee J_\lambda)$ does not hold.

In order to establish the condition that $V(L) = S(L)$, let us recall that a lattice L is called completely distributive if for any nonempty family $\{a_{i,j}\}_{i \in I, j \in J} \subseteq L$, whenever $\bigvee_{i \in I} \bigwedge_{j \in J} a_{i,j}$ and $\bigwedge_{f \in I^J} \bigvee_{i \in I} a_{i,f(i)}$ exist in L , then

$$\bigvee_{i \in I} \bigwedge_{j \in J} a_{i,j} = \bigwedge_{f \in I^J} \bigvee_{i \in I} a_{i,f(i)},$$

where I^J denotes the set of all maps from I to J .

Theorem 5.6. Let $L \in \mathbb{DL}$. The following conditions are equivalent:

- (1) $V(L) = S(L)$.
- (2) $\text{Ide}(L)$ is completely distributive.
- (3) $\text{Ide}(L)$ is α -distributive.

Proof. (1) \Rightarrow (2) Let $\{K_{i,j}\}_{i \in I, j \in J}$ be any nonempty family of ideals of L , and suppose that $\bigvee_{i \in I} \bigcap_{j \in J} K_{i,j}$ and $\bigcap_{f \in I^J} \bigvee_{i \in I} K_{i,f(i)}$ exist in $\text{Ide}(L)$. Write

$$A = \bigvee_{i \in I} \bigcap_{j \in J} K_{i,j}, \text{ and } B = \bigcap_{f \in I^J} \bigvee_{i \in I} K_{i,f(i)}.$$

Clearly, $A \subseteq B$. Since for any $I \in Ide(L)$, $I = \bigcap \{M \in V(L) \mid I \subseteq M\}$ and thus it suffices to show that for any $M \in V(L)$, if $M \supseteq A$ then $M \supseteq B$. Now, suppose $A \subseteq M$; then $\bigcap_{j \in J} K_{i,j} \subseteq M$ for any $i \in I$. By assumption, $M \in V(L) = S(L)$, so there exists some $j_i \in J$ such that $K_{i,j_i} \subseteq M$. Now let $f(i) = j_i$ for any $i \in I$; then $K_{i,f(i)} \subseteq M$ for any $i \in I$. It follows that $\bigvee_{i \in I} K_{i,f(i)} \subseteq M$. So we get $B \subseteq M$. Consequently, we obtain $A = B$.

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (1) Let $M \in V(L)$ and let $\{I_\lambda\}_{\lambda \in \Lambda}$ be any nonempty family of ideals of L with $|\Lambda| = \alpha$ such that $\bigcap_{\lambda \in \Lambda} I_\lambda \subseteq M$. Since R is dual α -distributive, we then have

$$M = M \vee \left(\bigcap_{\lambda \in \Lambda} I_\lambda \right) = \bigcap_{\lambda \in \Lambda} (M \vee I_\lambda).$$

So there exists some $\lambda_0 \in \Lambda$ such that $M \vee I_{\lambda_0} = M$, i.e., $I_{\lambda_0} \subseteq M$. So $M \in S(L)$. Therefore $V(L) = S(L)$. \square

At the end of this paper, we shall investigate decomposable lattices in which each nonzero element has only finitely many values.

Lemma 5.7. Let $L \in \mathbb{DL}$. If Q_1, Q_2, \dots, Q_n are mutually incomparable prime ideals of L and $a \notin Q_i$ for $i = 1, 2, \dots, n$, then there exist $a_i \in (\bigcap_{j \neq i} Q_j) \setminus Q_i$ such that $0 < a_i < a$ for $i = 1, 2, \dots, n$ and $a_i \wedge a_j = 0$ for $i \neq j$.

Proof. By induction on n . If $n = 2$ then pick $0 < x_1 \in Q_2 \setminus Q_1, 0 < x_2 \in Q_1 \setminus Q_2$. Clearly, $x_1 \parallel x_2$, so there exist $y_1, y_2 \in L$ such that

$$x_1 = y_1 \vee (x_1 \wedge x_2), x_2 = y_2 \vee (x_1 \wedge x_2) \text{ and } y_1 \wedge y_2 = 0.$$

Now, set $a_i = a \wedge y_i$ for $i = 1, 2$. Then $0 < a_1 \in Q_2 \setminus Q_1, 0 < a_2 \in Q_1 \setminus Q_2$ with $0 < a_i < a$ and $a_1 \wedge a_2 = 0$ for $i = 1, 2$.

Assume that the conclusion holds for the case $n - 1$. Now consider the case n . We divide the proof into three steps.

Step 1. For prime ideals Q_1, Q_2, \dots, Q_{n-1} , there exist $b_i \in (\bigcap_{1 \leq j \neq i \leq n-1} Q_j) \setminus Q_i$ such that $0 < b_i < a$ for $(i = 1, 2, \dots, n - 1)$ and $b_i \wedge b_j = 0$ for $i \neq j$.

Step 2. For prime ideals Q_2, Q_3, \dots, Q_n , there exist $c_i \in (\bigcap_{2 \leq j \neq i \leq n} Q_j) \setminus Q_i$ such that $0 < c_i < a$ for $i = 2, 3, \dots, n$ and $c_i \wedge c_j = 0$ for $i \neq j$.

Step 3. Set $a_i = b_i \wedge c_i$ for $i = 2, 3, \dots, n-1$. Clearly, $a_i \in (\bigcap_{1 \leq j \neq i \leq n} Q_j) \setminus Q_i$ with $0 < a_i < a$ for $i = 2, 3, \dots, n-1$ and $a_i \wedge a_j = 0$ for $i \neq j$.

Last, for prime ideals Q_1, Q_n , since $Q_1 \parallel Q_n$, pick $0 < f_1 \in Q_n \setminus Q_1, 0 < f_n \in Q_1 \setminus Q_n$ with $f_1 \wedge f_n = 0$. Set

$$a_1 = f_1 \wedge b_1, \quad a_n = f_n \wedge c_n.$$

Then $a_i \in (\bigcap_{j \neq i} Q_j) \setminus Q_i$ with $0 < a_i < a$ for $i = 1, 2, \dots, n$ and $a_i \wedge a_j = 0$ for $i \neq j$, which completes the proof. \square

Lemma 5.8. Let $L \in \mathbb{DL}$ and $0 < a \in L$. If a has only n values Q_1, Q_2, \dots, Q_n then $a = \bigvee_{i=1}^n a_i$ and each Q_i is the only value of a_i for $i = 1, 2, \dots, n$ and $a_i \wedge a_j = 0$ for $i \neq j$.

Proof. Clearly, Q_1, Q_2, \dots, Q_n are mutually incomparable prime ideals of L and $a \notin Q_i$ for $i = 1, 2, \dots, n$. By Lemma 5.7, there exist $a_i \in (\bigcap_{j \neq i} Q_j) \setminus Q_i$ such that $0 < a_i < a$ for $i = 1, 2, \dots, n$ and $a_i \wedge a_j = 0$ for $i \neq j$. Clearly, each Q_i is a value of a_i for $i = 1, 2, \dots, n$. Assume that a_i has another value, write Q_0 . Then $Q_0 \in \text{Val}(a)$, so there exists some $Q_j \in \text{Val}(a)$ with $j \neq i$ such that $Q_0 \subseteq Q_j$, and hence $a_j \in Q_0 \subseteq Q_j$, a contradiction.

Finally, we show that $a = \bigvee_{i=1}^n a_i$. Since $0 < a_i < a$ for $i = 1, 2, \dots, n$, we have $\bigvee_{i=1}^n a_i \leq a$. Assume that $\bigvee_{i=1}^n a_i < a$. Then $a \notin (\bigvee_{i=1}^n a_i]$. So there exists some $Q_i \in \text{Val}(a)$ such that $(\bigvee_{i=1}^n a_i] \subseteq Q_i$, and hence $a_i \in (\bigvee_{i=1}^n a_i] \subseteq Q_i$, a contradiction. So $a = \bigvee_{i=1}^n a_i$. \square

By Lemma 5.7 and Lemma 5.8, we have

Theorem 5.9. Let $L \in \mathbb{DL}$. Then the following conditions are equivalent:

- (1) Each nonzero element in L has only finitely many values.
- (2) For any $0 < a \in L$, $a = a_1 \vee a_2 \vee \dots \vee a_n$, where $a_i \wedge a_j = 0$ for $i \neq j$ and each a_i is special.

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